$J-$holomorphic Curves And Periodic Reeb Orbits *

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Abstract

We study the $J-$holomorphic curves in the symplectization of the contact manifolds and prove that there exists at least one periodic Reeb orbits in any closed contact manifold with any contact form by using the well-known Gromov's nonlinear Fredholm alternative for $J-$holomorphic curves. As a corollary, we give a complete solution on the well-known Weinstein conjecture.

1 Introduction and results

A contact structure on a manifold is a field of a tangent hyperplanes (contact hyperplanes) that is nondegenerate at any point. Locally such a field is defined as the field of zeros of a $1-$form λ, called a contact form. The nondegeneracy condition is $d\lambda$ is nondegenerate on the hyperplanes on which λ vanishes; equivalently, in $(2n-1)-$space:

$$\lambda \wedge (d\lambda)^{n-1} \neq 0$$

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The important example of contact manifold is the well-known projective cotangent bundles definded as follows:

Let $N = T^*M$ be the cotangent bundle of the smooth connected compact manifold M. N carries a canonical symplectic structure $\omega = -d\lambda$ where $\lambda = \sum_{i=1}^n y_i dx_i$ is the Liouville form on N, see [2, 6]. Let $P = PT^*M$ be the oriented projective cotangent bundle of M, i.e. $P = \cup_{x\in M} PT_x^*M$. It is well known that P carries a canonical contact structure induced by the Liouville form and the projection $\pi : T^*M \mapsto PT^*M$.

Let (Σ, λ) be a smooth closed oriented manifold of dimension $2n-1$ with a contact form λ. Associated to λ there are two important structures. First of all the so-called Reed vectorfield X_λ defined by

$$i_X\lambda \equiv 1, \quad i_X d\lambda \equiv 0$$

and secondly the contact structure $\xi = \xi_\lambda \mapsto \Sigma$ given by

$$\xi_\lambda = \ker(\lambda) \subset T\Sigma$$

by a result of Gray, [12] , the contact structure is very stable. In fact, if $(\lambda_t)_{t\in[0,1]}$ is a smooth arc of contact forms inducing the arc of contact structures $(\xi_t)_{t\in[0,1]}$, there exists a smooth arc $(\psi_t)_{t\in[0,1]}$ of diffeomorphisms with $\psi_0 = Id$, such that

$$T\Psi_t(\xi_0) = \xi_t \tag{1.1}$$

here it is important that Σ is compact. From (1.1) and the fact that $\Psi_0 = Id$ it follows immediately that there exists a smooth family of maps $[0,1] \times \Sigma \mapsto (0, \infty) : (t, m) \to f_t(m)$ such that

$$\Psi_t^*\lambda_t = f_t\lambda_0 \tag{1.2}$$

In contrast to the contact structure the dynamics of the Reeb vectorfield changes drastically under small perturbation and in general the flows associated to X_t and X_s for $t \neq s$ will not be conjugated, see[2, 6, 8, 15].

Let M be a Riemann manifold with Riemann metric, then it is well known that there exists a canonical contact structure in the unit sphere of its tangent bundle and the motion of geodesic line lifts to a geodesic flow on the unit sphere bundles. Therefore the closed orbit of geodesic flow or Reeb flow on the sphere bundle projects to a closed geodesics in the Riemann

manifolds, conversely the closed geodesic orbit lifts to a closed Reeb orbit. The classical work of Ljusternik and Fet states that every simply connected Riemannian manifold has at least one closed geodesics, this with the Cartan and Hadamard's results on non-simply closed Riemann manifold implies that any closed Riemann manifolds has a closed geodesics, i.e., the sphere bundle of a closed Riemann manifold with standard contact form carries at least one closed Reeb orbits which is a lift of closed geodesics of base manifold. Its proof depends on the classical minimax principle of Ljusternik and Schnirelman or minimalization of Hadamard and Cartan,[20]. In sympletic geometry, Gromov [13] introduces the global methods to proves the existences of symplectic fixed points or periodic orbits which depends on the nonlinear Fredholm alternative of $J-$holomorphic curves in the symplectic manifolds. In this paper we use the $J-$holomorphic curve's method to prove

Theorem 1.1 *Every closed contact manifold Σ with contact form λ carries at least one closed orbit.*

Note that Viterbo [30] first proved the above result for any contact manifolds Σ of restricted type in R^{2n} after Rabinowitz [26] and Weinstein [32, 33]. After Viterbo's work many results were obtained in [10, 15, 17, 16, 22, 23] etc by using variational method or Gromov's nonlinear Fredholm alternative, see survey paper [5]. Through $J-$holomorphic curves, especially, Hofer in [15] proved the following first striking results.

Corollary 1.1 *(Hofer)The three dimentional sphere with any contact form carries at least one closed Reeb orbit.*

Theorem 1.2 *(Ljusternik-Fet) Every simply connected closed Riemannian manifold has at least one closed geodesics.*

Therefore we get a new proof on the well-known Ljusternik-Fet Theorem without using the classical minimax principle, an alternative proof can be found in [25].

Theorem 1.3 *(Cartan-Hadamard) Every non-simply closed Riemannian manifold has at least one closed geodesics.*

Our method can not conclude that the geodesics is minimal.

Sketch of proofs: We work in the framework as in [13, 24]. In Section 2, we study the linear Cauchy-Riemann operator and sketch some basic properties. In section 3, first we construct a Lagrangian submanifold W under the assumption that there does not exists closed Reeb orbit in (Σ, λ); second, we study the space $\mathcal{D}(V, W)$ of contractible disks in manifold V with boundary in Lagrangian submanifold W and construct a Fredholm section of tangent bundle of $\mathcal{D}(V, W)$. In section 4, following [13, 24], we prove that the Fredholm section is not proper by using a special anti-holomorphic section as in [13, 24]. In section 5-6, we use a geometric argument to deduce the boundary C^0-estimates on W. In the final section, we use nonlinear Fredholm trick in [13, 24] to complete our proof.

Since the proofs in this paper are very difficult, we suggest the reader first read the Gromov's paper[13], Audin and Lafondaine's book[3], and Hummel's book[19].

Note 1.1 *The related problem with Weinstein conjecture(see[33]) is Arnold chord conjecture(see[1]) which was discussed in [1, 11] and finally solved in [24]. The generalized Arnold conjecture corresponding to Theorem 1.1 was also solved in similar method of this paper. These results was reported in the Second International Conference on Nonlinear Analysis, 14-19 June 1999, Tianjin, China; First* 3×3 *Canada-China Math Congress, Tsinghua University, Beijing, August 23-28,1999; Differential Geometry Seminar in Nankai Institute of Mathematics, Oct. 24-31, 2000; Symplectic Geometry Seminar In Nankai Institute of Math., Dec. 28-31, 2000; International conference on Symplectic geometry in Sichun Uni., June 24-July, 2001. Some technique part of proofs was carried in ICTP from August to October, 2001. The auther is deeply grateful to thank for the all inviters, especially to Professor Y. M. Long.*

2 Linear Fredholm Theory

For $100 < k < \infty$ consider the Hilbert space V_k consisting of all maps $u \in H^{k,2}(D, C \times C^n)$, such that $u(z) \in \{izR\} \times R^n \subset C \times C^n$ for almost all $z \in \partial D$. L_{k-1} denotes the usual Sobolev space $H_{k-1}(D, C \times C^n)$. We define

an operator $\bar{\partial} : V_k \mapsto L_{k-1}$ by

$$\bar{\partial} u = u_s + iu_t \tag{2.1}$$

where the coordinates on D are $(s,t) = s+it$, $D = \{z||z| \leq 1\}$. The following result is well known(see[34, p96,Th3.3.1]).

Proposition 2.1 *$\bar{\partial} : V_k \mapsto L_{k-1}$ is a surjective real linear Fredholm operator of index $n+3$. The kernel consists of $(a_0 + isz - \bar{a}_0 z^2, s_1, ..., s_n)$, $a_0 \in C$, $s, s_1, ..., s_n$.*

Let $(C^n, \sigma = -Im(\cdot,\cdot))$ be the standard symplectic space. We consider a real $n-$dimensional plane $R^n \subset C^n$. It is called Lagrangian if the skew-scalar product of any two vectors of R^n equals zero. For example, the plane $\{(p,q)|p=0\}$ and $\{(p,q)|q=0\}$ are two transversal Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of R^{2n} is called the Lagrangian-Grassmanian $\Lambda(n)$. One can prove that the fundamental group of $\Lambda(n)$ is free cyclic, i.e. $\pi_1(\Lambda(n)) = Z$. Next assume $(\Gamma(z))_{z\in\partial D}$ is a smooth map associating to a point $z \in \partial D$ a Lagrangian subspace $\Gamma(z)$ of C^n, i.e. $(\Gamma(z))_{z\in\partial D}$ defines a smooth curve α in the Lagrangian-Grassmanian manifold $\Lambda(n)$. Since $\pi_1(\Lambda(n)) = Z$, one have $[\alpha] = ke$, we call integer k the Maslov index of the closed curve α and denote it by $m(\Gamma)$ (see[2, p116-118]). Note that the Maslov index of the closed curve α is just the two times of the rotation numbers(see[2, p116-118] or [34, p96,Th3.3.1]).

Now let $z : S^1 \mapsto R \times R^n \subset C \times C^n$ be a smooth curve. Then it defines a constant loop α in Lagrangian-Grassmanian manifold $\Lambda(n+1)$. This loop defines the Maslov index $m(\alpha)$ of the map z which is easily seen to be zero.

Now Let (V, ω) be a symplectic manifold, $W \subset V$ a closed Lagrangian submanifold. Let $(\bar{V}, \bar{\omega}) = (D \times V, \omega_0 + \omega)$ and $\bar{W} = \partial D \times W$. Let $\bar{u} = (id, u) : (D, \partial D) \to (D \times V, \partial D \times W)$ be a smooth map homotopic to the map (id, u_0), here $u_0 : (D, \partial D) \to p \in W \subset V$. Then $\bar{u}^* TV$ is a symplectic vector bundle on D and $(\bar{u}|_{\partial D})^* T\bar{W}$ be a Lagrangian subbundle in $\bar{u}^* T\bar{V}|_{\partial D}$. Since $\bar{u} : (D, \partial D) \to (\bar{V}, \bar{W})$ is homotopic to $\bar{u}_0$, here$u_0(z) = (z, p)$, i.e., there exists a homotopy $h : [0,1] \times (D, \partial D) \to (\bar{V}, \bar{W})$ such that $h(0,z) = (z,p), h(1,z) = \bar{u}(z)$, we can take a trivialization of the symplectic vector bundle $h^* T\bar{V}$ on $[0,1] \times (D, \partial D)$ as

$$\Phi(h^* T\bar{V}) = [0,1] \times D \times C \times C^n$$

and

$$\Phi((h|_{[0,1]\times\partial D})^*T\bar{W}) \subset [0,1]\times S^1\times C\times C^n$$

Let

$$\pi_2 : [0,1]\times D\times C\times C^n \to C\times C^n$$

then

$$\tilde{h} : (s,z)\in[0,1]\times S^1 \to \pi_2\Phi(h|_{[0,1]\times\partial D})^*T\bar{W}|(s,z)\in\Lambda(n+1).$$

Lemma 2.1 *Let $\bar{u} : (D,\partial D)\to(\bar{V},\bar{W})$ be a C^k-map $(k\geq 1)$ as above. Then,*

$$m(\tilde{u}) = 2.$$

Proof. Since $\bar{u}$ is homotopic to $\bar{u}_0$ in $\bar{V}$ relative to $\bar{W}$, by the above argument we have a homotopy Φ_s of trivializations such that

$$\Phi_s(\bar{u}^*TV) = D\times C\times C^n$$

and

$$\Phi_s((\bar{u}|_{\partial D})^*T\bar{W}) \subset S^1\times C\times C^n$$

Moreover

$$\Phi_0(\bar{u}|_{\partial D})^*T\bar{W} = S^1\times izR\times R^n$$

So, the homotopy induces a homotopy $\tilde{h}$ in Lagrangian-Grassmanian manifold. Note that $m(\tilde{h}(0,\cdot)) = 0$. By the homotopy invariance of Maslov index, we know that $m(\tilde{u}|_{\partial D}) = 2$.

Consider the partial differential equation

$$\begin{aligned}
&\bar{\partial}\bar{u} + A(z)\bar{u} = 0 \ on\ D\\
&\bar{u}(z)\in\Gamma(z)(izR\times R^n)\ for\ z\in\partial D\\
&\Gamma(z)\in GL(2(n+1),R)\cap Sp(2(n+1))\\
&m(\Gamma) = 2
\end{aligned}\qquad (2.2)$$

For $100 < k < \infty$ consider the Banach space $\bar{V}_k$ consisting of all maps $u\in H^{k,2}(D,C^n)$ such that $u(z)\in\Gamma(z)$ for almost all $z\in\partial D$. Let L_{k-1} the usual Sobolev space $H_{k-1}(D,C\times C^n)$

Proposition 2.2 *$\bar{\partial} : \bar{V}_k \to L_{k-1}$ is a real linear Fredholm operator of index n+3.*

3 Nonlinear Fredholm Theory

3.1 Constructions of Lagrangian submanifolds

Let (Σ, λ) be a contact manifolds with contact form λ and X its Reeb vector field, then X integrates to a Reeb flow η_t for $t \in R^1$. Consider the form $d(e^a\lambda)$ at the point (a, σ) on the manifold $(R \times \Sigma)$, then one can check that $d(e^a\lambda)$ is a symplectic form on $R \times \Sigma$. Moreover One can check that

$$i_X(e^a\lambda) = e^a \tag{3.1}$$

$$i_X(d(e^a\lambda)) = -de^a \tag{3.2}$$

So, the symplectization of Reeb vector field X is the Hamilton vector field of e^a with respect to the symplectic form $d(e^a\lambda)$. Therefore the Reeb flow lifts to the Hamilton flow h_s on $R \times \Sigma$(see[2, 6, 8]).

Let

$$(V', \omega') = ((R \times \Sigma) \times (R \times \Sigma), d(e^a\lambda) \ominus d(e^b\lambda))$$

and

$$\mathcal{L} = \{((0, \sigma), (0, \sigma)) | (0, \sigma) \in R \times \Sigma\}.$$

Let

$$L' = \mathcal{L} \times R, L'_s = \mathcal{L} \times \{s\}.$$

Then define

$$\begin{aligned} &G' : L' \to V' \\ &G'(l') = G'(((\sigma, 0), (\sigma, 0)), s) = ((0, \sigma), (0, \eta_s(\sigma))) \end{aligned} \tag{3.3}$$

Then

$$W' = G'(L') = \{((0, \sigma), (0, \eta_s(\sigma))) | (0, \sigma) \in R \times \Sigma, s \in R\}$$

$$W'_s = G'(L'_s) = \{((0, \sigma), (0, \eta_s(\sigma))) | (0, \sigma) \in R \times \Sigma\}$$

for fixed $s \in R$.

Lemma 3.1 *There does not exist any Reeb closed orbit in (Σ, λ) if and only if $W'_s \cap W'_{s'}$ is empty for $s \neq s'$.*

Proof. First if there exists a closed Reeb orbit in (Σ, λ), i.e., there exists $\sigma_0 \in \Sigma$, $t_0 > 0$ such that $\sigma_0 = \eta_{t_0}(\sigma_0)$, then $((0, \sigma_0), (0, \sigma_0)) \in W'_0 \cap W'_{t_0}$. Second if there exists $s_0 \neq s'_0$ such that $W'_{s_0} \cap W'_{s'_0} \neq \emptyset$, i.e., there exists σ_0 such that

$$((0, \sigma_0), (0, \eta_{s_0}(\sigma_0)) = ((0, \sigma_0), (0, \eta_{s'_0}(\sigma_0)),$$

then $\eta_{(s_0 - s'_0)}(\sigma_0) = \sigma_0$, i.e., $\eta_t(\sigma_0)$ is a closed Reeb orbit.

Lemma 3.2 *If there does not exist any closed Reeb orbit in* (Σ, λ) *then there exists a smooth Lagrangian injective immersion* $G' : W' \to V'$ *with* $G'(((0, \sigma), (0, \sigma)), s) = ((0, \sigma), (0, \eta_s(\sigma)))$ *such that*

$$G'_{s_1, s_2} : \mathcal{L} \times (-s_1, s_2) \to V' \tag{3.4}$$

is a regular exact Lagrangian embedding for any finite real number s_1, s_2, *here we denote by* $W'(s_1, s_2) = G'_{s_1, s_2}(\mathcal{L} \times (s_1, s_2))$.

Proof. One check

$$G'^*((e^a \lambda - e^b \lambda)) = \lambda - \eta(\cdot, \cdot)^* \lambda = \lambda - (\eta_s^* \lambda + i_X \lambda ds) = -ds \tag{3.5}$$

since $\eta_s^* \lambda = \lambda$. This implies that G' is an exact Lagrangian embedding, this proves Lemma 3.2.

Now set

$$c(s, t) = \varepsilon t e^{-s^2} \tag{3.6}$$

$$\psi_0(s, t) = s e^{c(s,t)} c_s = -2 e^{(\varepsilon t e^{-s^2}) - s^2} \varepsilon t s^2 = \varepsilon \psi'_0 \tag{3.7}$$

here $\psi'_0(s, t) = -2ts^2 e^{(\varepsilon t e^{-s^2}) - s^2}$;

$$\psi_1(s, t) = \int_{-\infty}^{s} \psi_0(\tau, t) d\tau = \varepsilon \int_{-\infty}^{s} \psi'_0(\tau, t) d\tau = \varepsilon \psi'_1 \tag{3.8}$$

here $\psi'_1 = \int_{-\infty}^{s} \psi'_0$;

$$\psi(s, t) = \frac{\partial \psi_1}{\partial t} - s e^{c(s,t)} c_t = \varepsilon \psi' \tag{3.9}$$

here $\psi'(s, t) = \frac{\partial \psi_1}{\partial t} - s e^{\varepsilon t e^{-s^2}} e^{-s^2}$;

$$\Psi' = s e^{c(s,t)}; \tilde{l}' = -\psi(s, t) dt. \tag{3.10}$$

Now we construct an isotopy of Lagrangian injective immersions as follows:

$$
\begin{array}{l}
F' : \mathcal{L} \times R \times [0,1] \to (R \times \Sigma) \times (R \times \Sigma) \\
F'(((0,\sigma),(0,\sigma)),s,t) = ((c(s,t),\sigma),(c(s,t),\eta_s(\sigma))) \\
F'_t(((0,\sigma),(0,\sigma)),s) = F'(((0,\sigma),(0,\sigma)),s,t)
\end{array}
\tag{3.11}
$$

Lemma 3.3 *If there does not exist any closed Reeb orbit in* (Σ,λ) *then for the choice of* $c(s,t)$ *satisfying* $\int_0^s c(s,t)ds$ *or* $\int_s^0 c(s,t)ds$ *exists and is smooth on* (s,t)*,* F' *is an exact isotopy of Lagrangian embeddings. Moreover if* $c(s,0) \neq c(s,1)$*, then* $F'_0(\Sigma \times R) \cap F'_1(\Sigma \times R) = \emptyset$*.*

Proof. Let $F'_t = F'(\cdot,t) : \mathcal{L} \times R \to (R \times \Sigma) \times (R \times \Sigma)$. It is obvious that F'_t is an embedding. We check that

$$
\begin{array}{rcl}
F'^*(e^a\lambda \ominus e^b\lambda) & = & -e^{c(s,t)}ds \\
& = & -\{d(se^{c(s,t)}) - sde^{c(s,t)}\} \\
& = & -\{d(se^{c(s,t)}) - se^{c(s,t)}c_s ds - se^{c(s,t)}c_t dt\} \\
& = & -\{d(se^{c(s,t)}) - d_s\psi_1 - se^{c(s,t)}c_t dt\} \\
& = & -\{d((se^{c(s,t)}) - \psi_1) + \dfrac{\partial\psi_1}{\partial t}dt - se^{c(s,t)}c_t dt\} \\
& = & -\{d\Psi' + \dfrac{\partial\psi_1}{\partial t}dt - se^{c(s,t)}c_t dt\} \\
& = & -d\Psi' - \psi(s,t)dt \\
& = & -d\Psi' + \tilde{l}'
\end{array}
\tag{3.12}
$$

here ψ_i, ψ'_i, and $\tilde{l}', \Psi'$ as in (3.7-3.10).

Let (V',ω'), W' as above and $(V,\omega) = (V' \times C, \omega' \oplus \omega_0)$. As in [13, p330,2.3.$B'_3$](see also [3, p291-292]), we use figure eight trick invented by Gromov to construct a Lagrangian submanifold in V through the Lagrange isotopy F' in V'. Fix a positive $\delta < 1$ and take a C^∞-map $\rho : S^1 \to [0,1]$, where the circle S^1 ia parametrized by $\Theta \in [-1,1]$, such that the $\delta-$neighborhood I_0 of $0 \in S^1$ goes to $0 \in [0,1]$ and $\delta-$neighbourhood I_1 of $\pm 1 \in S^1$ goes $1 \in [0,1]$. Let

$$
\begin{array}{rcl}
\tilde{l} & = & \psi(s,\rho(\Theta))\rho'(\Theta)d\Theta \\
& = & \Phi d\Theta
\end{array}
\tag{3.13}
$$

be the pull-back of the form $\tilde{l}' = \psi(s,t)dt$ to $W' \times S^1$ under the map $(w',\Theta) \to (w',\rho(\Theta))$ and assume without loss of generality Φ vanishes on $W' \times (I_0 \cup I_1)$.

Next, consider a map α of the annulus $S^1 \times [\Phi_-, \Phi_+]$ into R^2, where Φ_- and Φ_+ are the lower and the upper bound of the fuction Φ correspondingly, such that

(i) The pull-back under α of the form $dx \wedge dy$ on R^2 equals $-d\Phi \wedge d\Theta$.

(ii) The map α is bijective on $I \times [\Phi_-, \Phi_+]$ where $I \subset S^1$ is some closed subset, such that $I \cup I_0 \cup I_1 = S^1$; furthermore, the origin $0 \in R^2$ is a unique double point of the map α on $S^1 \times 0$, that is

$$0 = \alpha(0,0) = \alpha(\pm 1, 0),$$

and α is injective on $S^1 = S^1 \times 0$ minus $\{0, \pm 1\}$.

(iii) The curve $S^1_0 = \alpha(S^1 \times 0) \subset R^2$ "bounds" zero area in R^2, that is $\int_{S^1_0} xdy = 0$, for the $1-$form xdy on R^2.

Proposition 3.1 *Let V', W' and F' as above. Then there exists an exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ given by $F(w', \Theta) = (F'(w', \rho(\Theta)), \alpha(\Theta, \Phi))$.*

Proof. We follow as in [13, 2.3B_3']. Now let $F^* : W' \times S^1 \to V' \times R^2$ be given by $(w', \Theta) \to (F'(w, \rho(\Theta)), \alpha(\Theta, \Phi))$. Then

$(i)'$ The pull-back under F^* of the form $\omega = \omega' + dx \wedge dy$ equals $d\tilde{l}^* - d\Phi \wedge d\Theta = 0$ on $W' \times S^1$.

$(ii)'$ The set of double points of F^* is $W_0' \cap W_1' \subset V' = V' \times 0 \subset V' \times R^2$.

$(iii)'$ If F^* has no double point then the Lagrangian submanifold $W = F^*(W' \times S^1) \subset (V' \times R^2, \omega' + dx \wedge dy)$ is exact if and only if $W_0' \subset V'$ is such.

This completes the proof of Proposition 3.1.

3.2 Formulation of Hilbert bundles

Let (Σ, λ) be a closed $(2n-1)-$ dimensional manifold with a contact form λ. Let $S\Sigma = R \times \Sigma$ and put $\xi = \ker(\lambda)$. Let J_λ' be an almost complex structure on $S\Sigma$ tamed by the symplectic form $d(e^a\lambda)$.

We define a metric g_λ on $S\Sigma = R \times \Sigma$ by

$$g_\lambda = d(e^a\lambda)(\cdot, J_\lambda \cdot) \tag{3.14}$$

which is adapted to J_λ and $d(e^a\lambda)$ but not complete.

In the following we denote by $(V', \omega') = ((R\times\Sigma)\times(R\times\Sigma), d(e^a\lambda_1 - e^b\lambda_2))$ and $(V,\omega) = (V'\times R^2, \omega' + dx\wedge dy)$ with the metric $g = g'\oplus g_0 = g_{\lambda_1}\oplus g_{\lambda_2}\oplus g_0$ induced by $\omega(\cdot, J\cdot)(J = J'\oplus i = J_{\lambda_1}\oplus(-J_{\lambda_2})\oplus i$ and $W\subset V$ a Lagrangian submanifold which was constructed in section 3.1.

Let $\bar{V} = D\times V$, then $\pi_1 : \bar{V}\to D$ be a symplectic vector bundle. Let $\bar{J}$ be an almost complex structure on $\bar{V}$ such that $\pi_1 : \bar{V}\to D$ is a holomorphic map and each fibre $\bar{V}_z = \pi_1(z)$ is a $\bar{J}$ complex submanifold. Let $H^k(D)$ be the space of H^k-maps from D to $\bar{V}$, here H^k represents Sobolev derivatives up to order k. Let $\bar{W} = \partial D\times W$, $\bar{p} = \{1\}\times p$, $W^{\pm} = \{\pm i\}\times W$ and

$$\mathcal{D}^k = \{\bar{u}\in H^k(D)|\bar{u}(x)\in\bar{W}\ a.e\ for\ x\in\partial D\ and\ \bar{u}(1) = \bar{p}, \bar{u}(\pm i)\in\{\pm i\}\times W\}$$

for $k\geq 100$.

Lemma 3.4 *Let W be a closed Lagrangian submanifold in V. Then, $\mathcal{D}^k$ is a pseudo-Hilbert manifold with the tangent bundle*

$$T\mathcal{D}^k = \bigcup_{\bar{u}\in\mathcal{D}^k}\Lambda^{k-1} \tag{3.15}$$

here

$$\Lambda^{k-1} = \{\bar{w}\in H^{k-1}(\bar{u}^*(T\bar{V})|\bar{w}(1) = 0, and\ \bar{w}(\pm i)\in T\bar{W}\}$$

Note 3.1 *Since W is not regular we know that $\mathcal{D}^k$ is in general complete, however it is enough for our purpose.*

Proof: See [3, p309-310] or follow step by step from [20, ch1].

Now we consider a section from $\mathcal{D}^k$ to $T\mathcal{D}^k$ follows as in [13, p327,2.2] or [13, p310], i.e., let $\bar{\partial} : \mathcal{D}^k\to T\mathcal{D}^k$ be the Cauchy-Riemmann section

$$\bar{\partial}\bar{u} = \frac{\partial\bar{u}}{\partial s} + J\frac{\partial\bar{u}}{\partial t} \tag{3.16}$$

for $\bar{u}\in\mathcal{D}^k$.

Theorem 3.1 *The Cauchy-Riemann section $\bar{\partial}$ defined in (3.16) is a Fredholm section of Index zero.*

Proof. According to the definition of the Fredholm section, we need to prove that $\bar{u} \in \mathcal{D}^k$, the linearization $D\bar{\partial}(\bar{u})$ of $\bar{\partial}$ at $\bar{u}$ is a linear Fredholm operator. Note that

$$D\bar{\partial}(\bar{u}) = D\bar{\partial}_{[\bar{u}]} \tag{3.17}$$

where

$$(D\bar{\partial}_{[\bar{u}]})v = \frac{\partial \bar{v}}{\partial s} + J\frac{\partial \bar{v}}{\partial t} + A(\bar{u})\bar{v} \tag{3.18}$$

with

$$\bar{v}|_{\partial D} \in (\bar{u}|_{\partial D})^* T\bar{W}$$

here $A(\bar{u})$ is $2n \times 2n$ matrix induced by the torsion of almost complex structure, see [13, p324,2.1] for the computation.

Observe that the linearization $D\bar{\partial}(\bar{u})$ of $\bar{\partial}$ at $\bar{u}$ is equivalent to the following Lagrangian boundary value problem

$$\begin{aligned} &\frac{\partial \bar{v}}{\partial s} + \bar{J}\frac{\partial \bar{v}}{\partial t} + A(\bar{u})\bar{v} = \bar{f},\ \bar{v} \in \Lambda^k(\bar{u}^* T\bar{V}) \\ &\bar{v}(t) \in T_{\bar{u}(t)}W, \quad t \in \partial D \end{aligned} \tag{3.19}$$

One can check that (3.19) defines a linear Fredholm operator. In fact, by proposition 2.2 and Lemma 2.1, since the operator $A(\bar{u})$ is a compact, we know that the operator $\bar{\partial}$ is a nonlinear Fredholm operator of the index zero.

Definition 3.1 *Let X be a Banach manifold and $P : Y \to X$ the Banach vector bundle. A Fredholm section $F : X \to Y$ is proper if $F^{-1}(0)$ is a compact set and is called generic if F intersects the zero section transversally, see [13, p327-328,2.2B].*

Definition 3.2 *$deg(F, y) = \sharp\{F^{-1}(0)\} mod 2$ is called the Fredholm degree of a Fredholm section (see[13, p327-328,2.2B]).*

Theorem 3.2 *Assum that $\bar{J} = i \oplus J$ on $\bar{V}$ and i is complex structure on D and J the almost complex structure on V. Assume that J is integrable at $p \in V$. Then the Fredholm section $F = \bar{\partial}_{\bar{J}} : \mathcal{D}^k \to T\mathcal{D}^k$ constructed in (3.16) has degree one, i.e.,*

$$deg(F, 0) = 1$$

Proof: We assume that $\bar{u} : D \mapsto \bar{V}$ be a $\bar{J}-$holomorphic disk with boundary $\bar{u}(\partial D) \subset \bar{W}$ and by the assumption that $\bar{u}$ is homotopic to the map $\bar{u}_1 = (id, p)$. Since almost complex structure $\bar{J}$ splits and is tamed by the symplectic form $\bar{\omega}$, by stokes formula, we conclude the second component $u : D \to V$ is a constant map. Because $u(1) = p$, We know that $F^{-1}(0) = (id, p)$. Next we show that the linearizatioon $DF_{(id,p)}$ of F at (id, p) is an isomorphism from $T_{(id,p)}\mathcal{D}^k$ to E. This is equivalent to solve the equations

$$\frac{\partial \bar{v}}{\partial s} + \bar{J}\frac{\partial \bar{v}}{\partial t} = f \tag{3.20}$$

$$\bar{v}|_{\partial D} \subset T_{(id,p)}\bar{W} \tag{3.21}$$

here $\bar{J} = i + J(p)$ since J is integrable at p. By Lemma 2.1, we know that $DF_{(id,p)}$ is an isomorphism. Therefore $deg(F, 0) = 1$.

4 Anti-holomorphic sections

In this section we construct a Fredholm section which is not proper as in [13, p329-330,2.3.B](see also [3, p315, 5.3]).

Let $(V', \omega') = (S\Sigma \times S\Sigma, d(e^a\lambda_1 - e^b\lambda_2))$ and $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$, W as in section3 and $J = J' \oplus i$, $g = g' \oplus g_0$, g_0 the standard metric on C.

Now let $c \in C$ be a non-zero vector. We consider c as an anti-holomorphic homomorphism $c : TD \to TV' \oplus TC$, i.e., $c(\frac{\partial}{\partial \bar{z}}) = (0, c \cdot \frac{\partial}{\partial z})$. Since the constant section c is not a section of the Hilbert bundle in section 3 due to c is not tangent to the Lagrangian submanifold W, we must modify it as follows:

Let c as in section 4.1, we define

$$c_{\chi,\delta}(z, v) = \begin{cases} c & \text{if } |z| \leq 1 - 2\delta, \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $c_{\chi,\delta}$, we obtain a smooth section c_δ satisfying

$$\begin{gathered} c_\delta(z, v) = \begin{cases} c & \text{if } |z| \leq 1 - 3\delta, \\ 0 & \text{if } |z| \geq 1 - \delta. \end{cases} \\ |c_\delta| \leq |c| \end{gathered} \tag{4.2}$$

for h small enough, for the convolution theory see [18, ch1,p16-17,Th1.3.1]. Then one can easily check that $\bar{c}_\delta = (0, 0, c_\delta)$ is an anti-holomorphic section tangent to $\bar{W}$.

Now we modify the almost conplex structure on the V. Let J_1, J_2 be the almost complex structures on V tamed by ω. Let $g_i = \omega(\cdot, J_i\cdot)$ the metrics by ω and J_i. We assume there exists a constant c_1 such that

$$c_1^{-1} g_1 \leq g_2 \leq c_1 g_1 \tag{4.3}$$

Let

$$J_{\chi,\delta}(z, v) = \begin{cases} i \oplus J_1 & \text{if } |z| \leq 1 - 2\delta, \\ i \oplus J_2 & \text{otherwise} \end{cases} \tag{4.4}$$

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $J_{\chi,\delta}$, we obtain a smooth section J_δ satisfying

$$J_\delta(z, v) = \begin{cases} i \oplus J_1 & \text{if } |z| \leq 1 - 3\delta, \\ i \oplus J_2 & \text{if } |z| \geq 1 - \delta. \end{cases} \tag{4.5}$$

for h small enough, for the convolution theory see [18, ch1,p16-17,Th1.3.1].

Now we get an almost conplex structure $\bar{J} = i \oplus J_\delta$ on the symplectic fibration $D \times V \to D$ such that $\pi_1 : D \times V \to D$ is a holomorphic fibration and $\pi_1^{-1}(z)$ is an almost complex submanifold. Let $g_\delta = \bar{\omega}(\cdot, \bar{J}\cdot)$, $\bar{g}_i = g_0 \oplus g_i$ be the metrics on $D \times V$, g_0 is metric on D. We assume there exists a constants c_2 such that

$$c_2^{-1} \bar{g}_i \leq g_\delta \leq c_2 g_i, i = 1, 2. \tag{4.6}$$

Now we consider the equations

$$\begin{aligned} &\bar{v} = (id, v) = (id, v', f) : D \to D \times V' \times C \\ &\bar{\partial}_{J_\delta} v = c_\delta \\ &\bar{\partial}_{J'} v' = 0, \bar{\partial} f = c_\delta \ on\ D_{1-2\delta} \\ &v|_{\partial D} : \partial D \to W \end{aligned} \tag{4.7}$$

here v homotopic to constant map $\{p\}$ relative to W. Note that $W \subset V \times B_R(0)$ for $2\pi R(\varepsilon)^2$, here $R(\varepsilon) \to 0$ as $\varepsilon \to 0$ and ε as in section 3.1.

Lemma 4.1 *Let $\bar{v} = (id, v)$ be the solutions of (4.7), then one has the following estimates*

$$E(v) = \{\int_D (g'(\frac{\partial v'}{\partial x}, J'\frac{\partial v'}{\partial x}) + g'(\frac{\partial v'}{\partial y}, J'\frac{\partial v'}{\partial y})$$
$$+g_0(\frac{\partial f}{\partial x}, i\frac{\partial f}{\partial x}) + g_0(\frac{\partial f}{\partial y}, i\frac{\partial f}{\partial y}))d\sigma\} \leq 4\pi R(\varepsilon)^2. \tag{4.8}$$

Proof: Since $v(z) = (v'(z), f(z))$ satisfy (4.7) and $v(z) = (v'(z), f(z)) \in V' \times C$ is homotopic to constant map $v_0 : D \to \{p\} \subset W$ in (V, W), by the Stokes formula

$$\int_D v^*(\omega' \oplus \omega_0) = 0 \tag{4.9}$$

Note that the metric g is adapted to the symplectic form ω and J, i.e.,

$$g = \omega(\cdot, J\cdot) \tag{4.10}$$

By the simple algebraic computation, we have

$$\int_D v^*\omega = \frac{1}{4}\int_{D^2}(|\partial v|^2 - |\bar{\partial} v|^2) = 0 \tag{4.11}$$

and

$$|\nabla v| = \frac{1}{2}(|\partial v|^2 + |\bar{\partial} v|^2 \tag{4.12}$$

Then

$$\begin{aligned} E(v) &= \int_D |\nabla v| \\ &= \int_D \{\frac{1}{2}(|\partial v|^2 + |\bar{\partial} v|^2)\})d\sigma \\ &= \int_D |c_\delta|^2_{\bar{g}} d\sigma \end{aligned} \tag{4.13}$$

By Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}\int_D \frac{\bar{\partial} f(\xi)}{\xi - z}d\xi \wedge d\bar{\xi} \tag{4.14}$$

Since f is smooth up to the boundary, we integrate the two sides on D_r for $r < 1$, one get

$$\int_{\partial D_r} f(z)dz \;=\; \int_{\partial D_r}\frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi - z}d\xi dz + \int_{\partial D_r}\frac{1}{2\pi i}\int_D \frac{\bar{\partial} f(\xi)}{\xi - z}d\xi \wedge d\bar{\xi}$$

$$
\begin{aligned}
&= 0 + \frac{1}{2\pi i}\int_D \int_{\partial D_r} \frac{\bar{\partial} f(\xi)}{\xi - z} dz d\xi \wedge d\bar{\xi} \\
&= \frac{1}{2\pi i}\int_D 2\pi i \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi} \qquad (4.15)
\end{aligned}
$$

Let $r \to 1$, we get

$$\int_{\partial D} f(z)dz = \int_D \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi} \qquad (4.16)$$

By the equations (4.7), one get

$$\bar{\partial} f = c \text{ on } D_{1-2\delta} \qquad (4.17)$$

So, we have

$$2\pi i(1-2\delta)c = \int_{\partial D} f(z)dz - \int_{D-D_{1-2\delta}} \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi} \qquad (4.18)$$

So,

$$
\begin{aligned}
|c| &\leq \frac{1}{2\pi(1-2\delta)} |\int_{\partial D} f(z)dz| + |\int_{D-D_{1-2\delta}} \bar{\partial} f(\xi) d\xi \wedge d\bar{\xi}| \\
&\leq \frac{1}{2\pi(1-2\delta)} 2\pi|diam(pr_2(W)) + c_1 c_2 |c|(\pi - \pi(1-2\delta)^2)) \qquad (4.19)
\end{aligned}
$$

Therefore, one has

$$|c| \leq c(\delta)R(\varepsilon) \qquad (4.20)$$

and

$$
\begin{aligned}
E(v) &= \pi \int_D |c_\delta|^2_{\bar{g}} \\
&= \pi c(\delta)^2 R(\varepsilon)^2. \qquad (4.21)
\end{aligned}
$$

This finishes the proof of Lemma.

Proposition 4.1 *For $|c| \geq 2c(\delta)R(\varepsilon)$, then the equations (4.7) has no solutions.*

Proof. By 4.20, it is obvious.

Theorem 4.1 *The Fredholm section $F_1 = \bar{\partial}_{\bar{J}} + \bar{c}_\delta : \mathcal{D}^k \to E$ is not proper.*

Proof. By the Proposition 4.1 and Theorem 3.2, it is obvious(see also [13, p330,2.3B_1] or [3, p316]).

5 $J-$holomorphic section

Recall that $W(-K,K) \subset W \subset V' \times R^2$ as in section 3. The Riemann metric g on $V' \times R^2$ induces a metric $g|W$.

Now let $c \in C$ be a non-zero vector and c_δ the induced anti-holomorphic section. We consider the nonlinear inhomogeneous equations (4.7) and transform it into $\bar{J}-$holomorphic map by considering its graph as in [13, p319,1.4.C] or [3, p312,Lemma5.2.3].

Denote by $Y^{(1)} \to D \times V$ the bundle of homomorphisms $T_s(D) \to T_v(V)$. If D and V are given the disk and the almost Kähler manifold, then we distinguish the subbundle $X^{(1)} \subset Y^{(1)}$ which consists of complex linear homomorphisms and we denote $\bar{X}^{(1)} \to D \times V$ the quotient bundle $Y^{(1)}/X^{(1)}$. Now, we assign to each C^1-map $v : D \to V$ the section $\bar{\partial} v$ of the bundle $\bar{X}^{(1)}$ over the graph $\Gamma_v \subset D \times V$ by composing the differential of v with the quotient homomorphism $Y^{(1)} \to \bar{X}^{(1)}$. If $c_\delta : D \times V \to \bar{X}$ is a H^k- section we write $\bar{\partial} v = c_\delta$ for the equation $\bar{\partial} v = c_\delta|\Gamma_v$.

Lemma 5.1 *(Gromov[13, 1.4.C′])There exists a unique almost complex structure J_g on $D \times V$(which also depends on the given structures in D and in V), such that the (germs of) $J_\delta-$holomorphic sections $v : D \to D \times V$ are exactly and only the solutions of the equations $\bar{\partial} v = c_\delta$. Furthermore, the fibres $z \times V \subset D \times V$ are $J_\delta-$holomorphic(i.e. the subbundles $T(z \times V) \subset T(D \times V)$ are $J_\delta-$complex) and the structure $J_\delta|z \times V$ equals the original structure on $V = z \times V$. Moreover J_δ is tamed by $k\omega_0 \oplus \omega$ for k large enough which is independent of δ.*

6 Gromov's C^0-convergence theorem

6.1 Analysis of Gromov's figure eight

Since $W' \subset S\Sigma \times S\Sigma$ is an exact Lagrangian submanifold and F'_t is an exact Lagrangian isotopy(see section 3.1). Now we carefully check the Gromov's construction of Lagrangian submanifold $W \subset V' \times R^2$ from the exact Lagrangian isotopy of W' in section 3.

Let $S^1 \subset T^*S^1$ be a zero section and $S^1 = \cup_{i=1}^4 S_i$ be a partition of the zero section S^1 such that $S_1 = I_0$, $S_3 = I_1$. Write $S^1 \setminus \{I_0 \cup I_1\} = I_2 \cup I_3$

and $I_0 = (-\delta, -\frac{5}{6}\delta] \cup (-\frac{5\delta}{6}, +\frac{5\delta}{6}) \cup [\frac{5\delta}{6}, \delta) = I_0^- \cup I_0' \cup I_0^+$, similarly $I_1 = (1-\delta, 1-\frac{5}{6}\delta] \cup (1-\frac{5\delta}{6}, 1+\frac{5\delta}{6}) \cup [1+\frac{5\delta}{6}, 1+\delta) = I_1^- \cup I_1' \cup I_1^+$. Let $S_2 = I_0^+ \cup I_2 \cup I_1^-$, $S_4 = I_1^+ \cup I_3 \cup I_0^+$. Moreover, we can assume that the double points of map α in Gromov's figure eight is contained in $(\bar{I}_0' \cup \bar{I}_1') \times [\Phi_-, \Phi_+]$, here $\bar{I}_0' = (-\frac{5\delta}{12}, +\frac{5\delta}{12})$ and $\bar{I}_1' = (1 - \frac{5\delta}{12}, 1 + \frac{5\delta}{12})$. Recall that $\alpha : (S^1 \times [5\Phi_-, 5\Phi_+]) \to R^2$ is an exact symplectic immersion, i.e., $\alpha^*(-ydx) - \Psi d\Theta = dh$, $h : T^*S^1 \to R$. By the construction of figure eight, we can assume that $\alpha_i' = \alpha|((S^1 \backslash I_i') \times [5\Phi_-, 5\Phi_+])$ is an embedding for $i = 0, 1$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset R^2$ and $Y_i = \alpha(S_i \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $\alpha_i = \alpha|Y_i(S^1 \times [5\Phi_-, 5\Phi_+])$. So, α_i puts the function h to the function $h_{i0} = \alpha_i^{-1*}h$ on Y_i. We extend the function h_{i0} to whole plane R^2. In the following we take the liouville form $\beta_{i0} = -ydx - dh_{i0}$ on R^2. This does not change the symplectic form $dx \wedge dy$ on R^2. But we have $\alpha_i^*\beta = \Phi d\Theta$ on $(S_i \times [5\Phi_-, 5\Phi_+])$ for $i = 1, 2, 3, 4$. Finally, note that

$$\begin{aligned} &F : W' \times S^1 \to V' \times R^2; \\ &F(w', \Theta) = (F'_{\rho(\Theta)}(w'), \alpha(\Theta, \Phi(w', \rho(\Theta))). \end{aligned} \tag{6.1}$$

Since $\rho(\Theta) = 0$ for $\Theta \in I_0$ and $\rho(\Theta) = 1$ for $\Theta \in I_1$, we know that $\Phi(w', \rho(\Theta)) = 0$ for $\Theta \in I_0 \cup I_1$. Therefore,

$$F(W' \times I_0) = W' \times \alpha(I_0); F(W' \times I_1) = W' \times \alpha(I_1). \tag{6.2}$$

6.2 Gromov's Schwartz lemma

In our proof we need a crucial tools, i.e., Gromov's Schwartz Lemma as in [13]. We first consider the case without boundary.

Proposition 6.1 *Let (V, J, μ) be as in section 4 and V_K the compact part of V. There exist constants ε_0 and C (depending only on the C^0- norm of μ and on the C^α norm of J and A_0) such that every $J-$holomorphic map of the unit disc to an ε_0-ball of V with center in V_K and area less than A_0 has its derivatives up to order $k + 1 + \alpha$ on $D_{\frac{1}{2}}(0)$ bounded by C.*

For a proof, see[13].

Now we consider the Gromov's Schwartz Lemma for $J-$holomorphic map with boundary in a closed Lagrangian submanifold as in [13].

Proposition 6.2 *Let (V, J, μ) as above and $L \subset V$ be a closed Lagrangian submanifold and V_K one compact part of V. There exist constants ε_0 and C(depending only on the C^0- norm of μ and on the C^α norm of J and K, A_0) such that every $J-$holomorphic map of the half unit disc D^+ to a ε_0-ball of V with boundary in L and area less than A_0 has its derivatives up to order $k+1+\alpha$ on $D^+_{\frac{1}{2}}(0)$ bounded by C.*

For a proof see [13].

Since in our case W is a non-compact Lagrangian submanifold, Proposition 6.2 can not be used directly but the proofs of Proposition 6.1-2 is still holds in our case.

Lemma 6.1 *Recall that $V = V' \times R^2$. Let (V, J, μ) as above and $W \subset V$ be as above and V_c the compact set in V. Let $\bar{V} = D \times V$, $\bar{W} = \partial D \times W$, and $\bar{V}_c = D \times V_c$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $Y_i = \alpha(S_i \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $\{X_j\}_{j=1}^q$ be a Darboux covering of Σ and $V'_{ij} = (R \times X_i) \times (R \times X_j)$. Let $\partial D = S^{1+} \cup S^{1-}$. There exist constant c_0 such that every $J-$holomorphic map v of the half unit disc D^+ to the $D \times V'_j \times R^2$ with its boundary $v((-1, 1)) \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i) \subset \bar{W}, i = 1, .., 4$ has*

$$area(v(D^+)) \leq c_0 l^2(v(\partial' D^+)). \tag{6.3}$$

here $\partial' D^+ = \partial D \setminus [-1, 1]$ and $l(v(\partial' D^+)) = length(v(\partial' D^+))$.

Proof. Let $\bar{W}_{i\pm} = S^{1\pm} \times F(W' \times S_i)$. Let $v = (v_1, v_2) : D^+ \to \bar{V} = D \times V$ be the $J-$holomorphic map with $v(\partial D^+) \subset \bar{W}_{i\pm} \subset \partial D \times W$, then

$$\begin{aligned}
area(v) &= \int_{D^+} v^* d(\alpha_0 \oplus \alpha) \\
&= \int_{D^+} dv^*(\alpha_0 \oplus \alpha) \\
&= \int_{\partial D^+} v^*(\alpha_0 \oplus \alpha) \\
&= \int_{\partial D^+} v_1^* \alpha_0 + \int_{\partial D^+} v_2^* \alpha \\
&= \int_{\partial' D^+ \cup [-1,+1]} v_1^* \alpha_0 + \int_{\partial' D^+ \cup [-1,+1]} v_2^*(e^a \lambda - y dx - dh_{i0}) \\
&= \int_{\partial' D^+ \cup [-1,+1]} v_1^* \alpha_0 + \int_{\partial' D^+} v_2^*(e^a \lambda - y dx - dh_{i0}) + B_1, \quad (6.4)
\end{aligned}$$

here $B_1 = \int_{[-1,+1]} v_2{}^*(-d\Psi')$. Now take a zig-zag curve C in $V'_j \times Y_i$ connecting $v_2(-1)$ and $v_2(+1)$ such that

$$\begin{aligned} \int_C (e^a\lambda + ydx) &= B_1 \\ length(C) &\leq k_1 length(v_2(\partial' D^+)) \end{aligned} \tag{6.5}$$

Now take a minimal surface M in $V'_{ij} \times R^2$ bounded by $v_2(\partial' D^+) \cup C$, then by the isoperimetric ineqality(see[[14, p283]), we get

$$\begin{aligned} area(M) &\leq m_1 length(C + v_2(\partial' D^+))^2 \\ &\leq m_2 length(v_2(\partial' D^+))^2, \end{aligned} \tag{6.6}$$

here we use the (6.5).

Since $area(M) \geq \int_M \omega$ and $\int_M \omega = \int_{D^+} v_2^* \omega = area(v)$, this proves the lemma.

Lemma 6.2 *Let v as in Lemma 6.1, then we have*

$$area(v(D^+) \geq c_0 (dist(v(0), v(\partial' D^+)))^2, \tag{6.7}$$

here c_0 depends only on $\Sigma, J, \omega, ...,$etc, not on v.

Proof. By the standard argument as in [3, p79].

The following estimates is a crucial step in our proof.

Lemma 6.3 *Recall that $V = V' \times R^2$. Let (V, J, μ) as above and $W \subset V$ be as above and V_c the compact set in V. Let $\bar{V} = D \times V$, $\bar{W} = \partial D \times W$, and $\bar{V}_c = D \times V_c$. Let $Y = \alpha(S^1 \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $Y_i = \alpha(S_i \times [5\Phi_-, 5\Phi_+]) \subset R^2$. Let $\partial D = S^{1+} \cup S^{1-}$. There exist constant c_0 such that every $J-$holomorphic map v of the half unit disc D^+ to the $D \times V' \times R^2$ with its boundary $v((-1, 1)) \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i) \subset \bar{W}, i = 1, .., 4$ has*

$$area(v(D^+)) \leq c_0 l^2(v(\partial' D^+)). \tag{6.8}$$

here $\partial' D^+ = \partial D \setminus [-1, 1]$ and $l(v(\partial' D^+)) = length(v(\partial' D^+))$.

Proof. We first assume that ε in section 3.1 is small enough. Let l_0 is a constant small enough. If $length(\partial' D^+) \geq l_0$, then Lemma 6.3 holds. If $length(\partial' D^+) \leq l_0$ and $v(D^+) \subset D \times V'_{ij} \times R^2$, then Lemma6.3 reduces to Lemma6.1. If $length(\partial' D^+) \leq l_0$ and $v(D^+) \bar{\subset} D \times V'_{ij} \times R^2$, then Lemma6.2 imples $area(v) \geq \tau_0 > 100\pi R(\varepsilon)^2$, this is a contradiction. Therefore we proved the lemma.

Proposition 6.3 *Let (V, J, μ) and $W \subset V$ be as in section 4 and V_K the compact part of V. Let $\bar{V}$, $\bar{V}_K$ and $\bar{W}$ as section 5.1. There exist constants ε_0 (depending only on the C^0- norm of μ and on the C^α norm of J) and C(depending only on the C^0 norm of μ and on the $C^{k+\alpha}$ norm of J) such that every $J-$holomorphic map of the half unit disc D^+ to the $D \times V' \times R^2$ with its boundary $v((-1, 1)) \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i) \subset \bar{W}, i = 1, .., 4$ has its derivatives up to order $k + 1 + \alpha$ on $D^+_{\frac{1}{2}}(0)$ bounded by C.*

Proof. One uses Lemma 6.3 and Gromov's proof on Schwartz lemma to yield proposition 6.3.

6.3 Removal singularity of $J-$curves

In our proof we need another crucial tools, i.e., Gromov's removal singularity theorem[13]. We first consider the case without boundary.

Proposition 6.4 *Let (V, J, μ) be as in section 4 and V_K the compact part of V. If $v : D \setminus \{0\} \to V_K$ be a $J-$holomorphic disk with bounded energy and bounded image, then v extends to a $J-$holomorphic map from the unit disc D to V_K.*

For a proof, see[13].

Now we consider the Gromov's removal singularity theorem for $J-$holomorphic map with boundary in a closed Lagrangian submanifold as in [13].

Proposition 6.5 *Let (V, J, μ) as above and $L \subset V$ be a closed Lagrangian submanifold and V_K one compact part of V. If $v : (D^+ \setminus \{0\}, \partial'' D^+ \setminus \{0\}) \to (V_K, L)$ be a $J-$holomorphic half-disk with bounded energy and bounded image, then v extends to a $J-$holomorphic map from the half unit disc $(D^+, \partial'' D^+)$ to (V_K, L).*

For a proof see [13].

Proposition 6.6 *Let (V, J, μ) and $W \subset V$ be as in section 4 and V_c the compact set in V. Let $\bar{V} = D \times V$, $\bar{W} = \partial D \times W$, and $\bar{V}_c = D \times V_c$. Then every $J-$holomorphic map v of the half unit disc $D^+ \setminus \{0\}$ to the $\bar{V}$ with center in $\bar{V}_c$ and its boundary $v((-1,1) \setminus \{0\}) \subset (S^{1\pm}) \times F(\mathcal{L} \times [-K, K] \times S_i) \subset \bar{W}$ and*

$$area(v(D^+ \setminus \{0\})) \leq E \tag{6.9}$$

extends to a $J-$holomorphic map $\tilde{v} : (D^+, \partial'' D) \to (\bar{V}_c, \bar{W})$.

Proof. This is ordinary Gromov's removal singularity theorem by $K-$assumption.

6.4 C^0-Convergence Theorem

We now recall that the well-known Gromov's compactness theorem for cusp's curves for the compact symplectic manifolds with closed Lagrangian submanifolds in it. For reader's convenience, we first recall the "weak-convergence" for closed curves.

Cusp-curves. Take a system of disjoint simple closed curves γ_i in a closed surface S for $i = 1, ..., k$, and denote by S^0 the surface obtained from $S \setminus \cup_{i=1}^{k} \gamma_i$. Denote by $\bar{S}$ the space obtained from S by shrinking every γ_i to a single point and observe the obvious map $\alpha : S^0 \to \bar{S}$ gluing pairs of points s'_i and s''_i in S^0, such that $\bar{s}_i = \alpha(s'_i) = \alpha(s''_i) \in \bar{S}$ are singular (or cuspidal) points in $\bar{S}$(see[13]).

An almost complex structure in $\bar{S}$ by definition is that in S^0. A continuous map $\beta : \bar{S} \to V$ is called a (parametrized $J-$holomorphic) cusp-curve in V if the composed map $\beta \circ \alpha : S^0 \to V$ is holomorphic.

Weak convergence. A sequence of closed $J-$curves $C_j \subset V$ is said to weakly converge to a cusp-curve $\bar{C} \subset V$ if the following four conditions are satisfied

(i) all curves C_j are parametrized by a fixed surface S whose almost complex structure depends on j, say $C_j = f_j(S)$ for some holomorphic maps

$$f_j : (S, J_j) \to (V, J)$$

(ii) There are disjoint simple closed curves $\gamma_i \in S$, $i = 1, ..., k$, such that $\bar{C} = \bar{f}(\bar{S})$ for a map $\bar{f} : \bar{S} \to V$ which is holomorphic for some almost complex structure $\bar{J}$ on $\bar{S}$.

(iii) The structures J_j uniformly $C^\infty-$converge to $\bar{J}$ on compact subsets in $S \setminus \cup_{i=1}^k \gamma_i$.

(iv) The maps f_j uniformly $C^\infty-$converge to $\bar{f}$ on compact subsets in $S \setminus \cup_{i=1}^k \gamma_i$. Moreover, f_j uniformly C^0-converge on entire S to the composed map $S \to \bar{S} \xrightarrow{\bar{f}} V$. Furthermore,

$$Area_\mu f_j(S) \to Area_\mu \bar{f}(\bar{S}) \ for\ j \to \infty,$$

where μ is a Riemannian metric in V and where the area is counted with the geometric multiplicity(see[13]).

Gromov's Compactness theorem for closed curves. Let C_j be a sequence of closed $J-$curves of a fixed genus in a compact manifold (V, J, μ). If the areas of C_j are uniformly bounded,

$$Area_\mu \leq A,\ j = 1, ..,$$

then some subsequence weakly converges to a cusp-curve $\bar{C}$ in V.

Cusp-curves with boundary. Let T be a compact complex manifold with boundary of dimension 1(i.e., it has an atlas of holomorphic charts onto open subsets of C or of a closed half plane). Its double is a compact Riemann surface S with a natureal anti-holomorphic involution τ which exchanges T and $S \setminus T$ while fixing the boundary ∂T. IF$f : T \to V$ is a continous map, holomorphic in the interior of T, it is convenient to extend f to S by

$$f = f \circ \tau$$

Take a totally real submanifold $W \subset (V, J)$ and consider compact holomorphic curves $C \subset V$ with boundaries, $(\bar{C}, \partial\bar{C}) \subset (V, W)$, which are, topologically speaking, obtained by shrinking to points some (short) closed loops in C and also some (short) segments in C between boundary points. This is seen by looking on the double $C \cup_{\partial C} C$.

Gromov's Compactness theorem for curves with boundary. Let V be a closed Riemannian manifold, W a totally real closed submanifold of

V. Let C_j be a sequence of $J-$curves with boundary in W of a fixed genus in a compact manifold (V, J, μ). If the areas of C_j are uniformly bounded,

$$Area_\mu \leq A, \;\; j = 1, ..,$$

then some subsequence weakly converges to a cusp-curve $\bar{C}$ in V.

The proofs of Gromov's compactness theorem can found in [**?**, 13]. In our case the Lagrangian submanifold W is not compact, Gromov's compactness theorem can not be applied directly but its proof is still effective since the W has the special geometry. In the following we modify Gromov's proof to prove the C^0-compactness theorem in our case.

Now we state the C^0-convergence theorem in our case.

Theorem 6.1 *Let (V, J, ω, μ) and W as in section4. Let C_j be a sequence of $\bar{J}_\delta-$holomorphic section $v_j = (id, ((a_j^1, u_j^1), (a_j^2, u_j^2), f_j)) : D \to D \times V$ with $v_j : \partial D \to \partial D \times W$ and $v_j(1) = (1, p) \in \partial D \times W$. constructed from section 4. Then the areas of C_j are uniformly bounded,i.e.,*

$$Area_\mu(C_j) \leq A, \;\; j = 1, ..,$$

and some subsequence weakly converges to a cusp-section $\bar{C}$ in V (see[3, 13]).

Proof. We follow the proofs in [13]. Write $v_j = (id, (a_j^1, u_j^1), (a_j^2, u_j^2), f_j))$ then $|a_{ij}^2| \leq a_0$ by the ordinary Monotone inequality of minimal surface without boundary, see following Proposition 7.1. Similarly $|f_j| \leq R_1$ by using the fact $f_j(\partial D)$ is bounded in $B_{R_1}(0)$ and $\int_D |\nabla f_j| \leq 4\pi R^2$ via monotone inequality for minimal surfaces. So, we assume that $v_j(D) \subset V_c$ for a compact set V_c.

1. *Removal of a net.*

1a. Let $\bar{V} = D \times V$ and v_j be regular curves. First we study induced metrics μ_j in v_j. We apply the ordinary monotone inequality for minimal surfaces without boundary to small concentric balls $B_\varepsilon \subset (A_j, \mu_j)$ for $0 < \varepsilon \leq \varepsilon_0$ and conclude by the standard argument to the inequality

$$Area(B_\varepsilon) \geq \varepsilon^2, \;\; for \; \varepsilon \leq \varepsilon_0;$$

Using this we easily find a interior $\varepsilon-$net $F_j \subset (v_j, \mu_j)$ containing N points for a fixed integer $N = (\bar{V}, \bar{J}, \mu)$, such that every topological annulus $A \subset v_j \setminus F_j$ satisfies

$$Diam_\mu A \leq 10 length_\mu \partial A. \tag{6.10}$$

Furthermore, let A be conformally equivalent to the cylinder $S^1 \times [0, l]$ where S^1 is the circle of the unit length, and let $S^1_t \subset A$ be the curve in A corresponding to the circle $S^1 \times t$for $t \in [0, l]$. Then obviously

$$\int_a^b (lengthS^1_t)^2 dt \leq Area(A) \leq C_5. \tag{6.11}$$

for all $[a, b] \subset [0, l]$. Hence, the annulus $A_t \subset A$ between the curves S^1_t and S^1_{l-t} satisfies

$$diam_\mu A_t \leq 20(\frac{C_5}{t}) \tag{6.12}$$

for all $t \in [0, l]$.

1b. We consider the sets $\partial v_j \cap ((S^{1\pm}) \times F(W' \times I_i^{\pm})), i = 0, 1$. By the construction of Gromov's figure eight, there exists a finite components, denote it by

$$\partial v_j \cap ((S^{1\pm}) \times F(\mathcal{L} \times R \times I_i^{\pm})) = \{\bar{\gamma}^k_{ij}\}, i = 0, 1. \tag{6.13}$$

we choose one point in $\bar{\gamma}^k_{ij}$ as a boundary puncture point in ∂v_j for each i, k.

Consider the concentric ε half-disks or quadrature $B_\varepsilon(p)$ with center p on $\bar{\gamma}^k_{ij}$, then

$$Area(B_\varepsilon(p)) \geq \tau_0 \tag{6.14}$$

Since $Area(v_j) \leq E_0$, there exists a uniform finite puncture points.

So, we find a boundary net $G_j \subset \partial v_j$ containing N_1 points for a fixed integer $N_1(\bar{V}, \bar{J}, \mu)$, such that every topological quadrature or half annulus $B \subset v_j \setminus \{F_j, G_j\}$ satisfies

$$\partial'' B = \partial B \cap \bar{W} \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i), i = 1, 2, 3, 4. \tag{6.15}$$

2. *Poincare's metrics*. 2a. Now, let μ^*_j be a metric of constant curvature -1 in $v_j(D) \setminus F_j \cup G_j$ conformally equivalent to μ_j. Then for every μ^*_j-ball B_ρ in $v_j \setminus F_j \cup G_j$ of radius $\rho \leq 0.1$, there exists an annulus A contained in $v_j \setminus F_j \cup G_j$ such that $B_\rho \subset A_t$ for $t = 0.01|log|$(see Lemma 3.2.2in [**?**, chVIII]). This implies with (6.3) the uniform continuity of the (inclusion) maps $(v_j \setminus F_j, \mu^*_j) \to (\bar{V}, \bar{\mu})$, and hence a uniform bound on the r^{th} order differentials for every $r = 0, 1, 2, ...$

2b. Similarly, for every μ^*_j-half ball B^+_ρ in $v_j \setminus F_j \cup G_j$ of radius $\rho \leq 0.1$, there exists a half annulus or quadrature B contained in $v_j \setminus F_j \cup G_j$ such

that $B_\rho^+ \subset B$ with

$$\partial'' B = \partial B \cap \bar{W} \subset (S^{1\pm}) \times F(\mathcal{L} \times R \times S_i), i = 1, 2, 3, 4. \quad (6.16)$$

Then, by Gromov's Schwartz Lemma, i.e., Proposition 6.1-6.3 implies the uniform bound on the r^{th} order differentials for every $r = 0, 1, 2, ...$

3. *Convergence of metrics.* Next, by the standard (and obvious) properties of hyperbolic surfaces there is a subsequence(see[3]), which is still denoted by v_j, such that

(a). There exist k closed geodesics or geodesic arcs with boundaries in $\partial v_j \setminus F_j$, say

$$\gamma_i^j \subset (v_j \setminus F_j, \mu_j^*), i = 1, ..., k, j = 1, 2, ...,$$

whose μ_j^*-length converges to zero as $j \to \infty$, where k is a fixed integer.

(b). There exist k closed curves or geodesic arcs with boundaries in ∂S of a fixed surface, say γ_j in S, and an almost complex structure $\bar{J}$ on the corresponding (singular) surface $\bar{S}$, such that the almost complex structure J_j on $v_j \setminus F_j$ induced from (V, J) $C^\infty-$converge to $\bar{J}$ outside $\cup_{j=1}^k \gamma_j$. Namely, there exist continuous maps $g^j : v_j \to \bar{S}$ which are homeomorphisms outside the geodesics γ_i^j, which pinch these geodesics to the corresponding singular points of $\bar{S}$(that are the images of γ_i) and which send F_j to a fixed subset F in the nonsingular locus of $\bar{S}$. Now, the convergence $J_j \to \bar{J}$ is understood as the uniform $C^\infty-$convergence $g_*^j(J_j) \to \bar{J}$ on the compact subsets in the non-singular locus $\bar{S}^*$ of $\bar{S}$ which is identified with $S \setminus \cup_{i=1}^k \gamma_i$.

4. C^0-*interior convergence.* The limit cusp-curve $\bar{v} : \bar{S}^* \to \bar{V}$, that is a holomorphic map which is constructed by first taking the maps

$$\bar{v}_j = (g_j)^{-1} : S \setminus \cup_{i=1}^k \gamma_i \to \bar{V}$$

Near the nodes of $\bar{S}$ including interior nodes and boundary nodes, by the properties of hyperbolic metric μ^* on $\bar{S}$, the neighbourhoods of interior nodes are corresponding to the annulis of the geodesic cycles. By the reparametrization of v_j, called $\bar{v}_j$ which is defined on S and extends the maps $\bar{v}_j : S \to S_j \to V$(see[3, 13]). Now let $\{z_i | i = 1, ..., n\}$ be the interior nodes of $\bar{S}$. Then the arguments in [3, 13] yield the C^0-interir convergece near z_i.

5. C^0-*boundary convergence.* Now it is possible that the boundary of the cusp curve $\bar{v}$ does not remain in $\bar{W}$. Write $\bar{v}(z) = (h, ((a_1, u_1), (a_2, u_2)), f)(z)$,

here $h(z) = z$ or $h(z) \equiv z_i$, $i = 1, ..., n$, z_i is cusp-point or bubble point. We can assume that $\bar{p} = (1, p) \in \bar{v}_n$ is a puncture boundary point. Let $\bar{v}_1$ be the component of $\bar{v}$ which through the point $\bar{p}$. Let $D = \{z | z = re^{i\theta}, 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. We assume that $\bar{v}_1 : D \setminus \{e^{i\theta_i}\}_{i=1}^k \to V_c$, here $e^{i\theta_i}$ is node or puncture point. Near $e^{i\theta_i}$, we take a small disk D_i in D containing only one puncture or node point $e^{i\theta_i}$. By the reparametrization and the convergence procedure, we can assume that $\bar{v}_{1i} = (\bar{v}_1 | D_i)$ as a map from $D^+ \setminus \{0\} \to V_c$ with $\bar{v}_1([-1, 1] \setminus \{0\}) \subset S^1 \times F(W' \times S^1)$ and $area(\bar{v}_{1i}) \le a_0$, a_0 small enough. Since $Area(\bar{v}_{1i}) \le a_0$, there exist curves c_k near 0 such that $l(\bar{v}_{1i}(c_k)) \le \delta_1$. By the construction of convergence, we can assume that $l(\bar{v}_n(c_k)) \le 2\delta_1$. If $\bar{v}_{1i}(\partial c_k) \subset (S^1) \times F(\mathcal{L} \times [-N_0, N_0] \times S^1)$, we have $\bar{v}_n(\partial c_k) \subset (S^1) \times F(\mathcal{L} \times [-2N_0, 2N_0] \times S^1)$ for n large enough. Now $\bar{v}_n(c_k)$ cuts $\bar{v}_n(D)$ as two parts, one part corresponds to $\bar{v}_{1i}$, say $\bar{u}_n(D)$. Then $area(\bar{u}_n(D)) = area(u_{n1}) + |\Psi'(u_{n2}(c_k^1)) - \Psi'(u_{n2}(c_k^2))|$, here $\partial c_k = \{c_k^1, c_k^2\}$. Then by the proof of Lemma6.1-6.3, we know that $\bar{u}_n(\partial D \setminus c_k) \subset (S^1) \times F(\mathcal{L} \times [-100N_0, 100N_0] \times S^1)$. So, $\bar{v}_{1i}([-1, 1] \setminus \{0\}) \subset S^1 \times F(\mathcal{L} \times [-100N_0, 100N_0] \times S^1)$. By proposition 6.6, one singularity of $\bar{v}_1$ is deleted. We repeat this procedure, we proved that $\bar{v}_1$ is extended to whole D. So, the boundary node or puncture points of $\bar{v}$ are removed. Then by choosing the sub-sub-sequences of μ_j^* and $\bar{v}_j$, we know that $\bar{v}_j$ converges to $\bar{v}$ in C^0 near the boundary node or puncture point. This proved the C^0-boundary convergence. Since $\bar{v}_j(1) = \bar{p}$, $\bar{p} \in \bar{v}(\partial D)$, $\bar{v}(\partial D) \subset \bar{W}$.

6. *Convergence of area.* Finally by the C^0-convergence and $area(v_j) = \int_D v_j^* \bar{\omega}$, one easily deduces

$$area(v(S)) = \lim_{j \to \infty} (v_j(S_j)).$$

6.5 Bounded image of $J-$holomorphic curves in W

Proposition 6.7 *Let v be the solutions of equations (4.16), then*

$$d_W(p, v(\partial D^2)) = max\{d_W(p, q) | q \in f(\partial D^2)\} \le d_0 < +\infty$$

Proof. It follows directly from Gromov's C^0-convergence theorem.

7 Proof of Theorem 1.1

Proposition 7.1 *If $J-$holomorphic curves $C \subset \bar{V}$ with boundary*

$$\partial C \subset D^2 \times ([0, \varepsilon] \times \Sigma) \times ([0, \varepsilon] \times \Sigma) \times R^2$$

and

$$C \cap (D^2 \times (\{-3\} \times \Sigma) \times (R \times \Sigma) \times R^2) \neq \emptyset$$

or

$$C \cap (D^2 \times (R \times \Sigma) \times (\{-3\} \times \Sigma) \times R^2) \neq \emptyset$$

Then

$$area(C) \geq 2l_0.$$

Proof. It is obvious by monotone inequality argument for minimal surfaces.

Note 7.1 *we first observe that any $J-$holomorphic curves with boundary in $R^+ \times \Sigma$ meet the hypersurface $\{-3\} \times \Sigma$ has energy at least $2l_0$, so we take ε small enough such that the Gromov's figure eight contained in $B_{R(\varepsilon)} \subset C$ for ε small enough and the energy of solutions in section 4 is smaller than l_0. we specify the constant a_0, ε in section 3.1-3 such that the above conditions satisfied.*

Theorem 7.1 *There exists a non-constant $J-$holomorphic map $u : (D, \partial D) \to (V' \times C, W)$ with $E(u) \leq 4\pi R(\varepsilon)^2$ for ε small enough such that $4\pi R(\varepsilon)^2 \leq l_0$.*

Proof. By Proposition 5.1, we know that the image $\bar{v}(D)$ of solutions of equations (4.7) remains a bounded or compact part of the non-compact Lagrangian submanifold W. Then, all arguments in [3, 13] for the case W is closed in $S\Sigma \times S\Sigma \times R^2$ can be extended to our case, especially Gromov's C^0-converngence theorem holds. But the results in section 4 shows the solutions of equations (4.7) must denegerate to a cusp curves, i.e., we obtain a Sacks-Uhlenbeck-Gromov's bubble, i.e., $J-$holomorphic sphere or disk with boundary in W, the exactness of ω rules out the possibility of $J-$holomorphic sphere. For the more detail, see the proof of Theorem 2.3.B in [13].

Proof of Theorem 1.1. If (Σ, λ) has not closed Reeb orbit, then we can construct a Lagrangian submanifold W in $V = V' \times C$, see section 3. Then as

in section 4, we construct an anti-holomorphic section c and for large vector $c \in C$ we know that the nonlinear Fredholm section or Cauchy-Riemann section has no solution, this implies that the section is non-proper, see section 4. The non-properness of the section and the Gromov's compactness theorem in section 6 implies the existences of the cusp-curves. So, we must have the $J-$holomorphic sphere or $J-$holomorphic disk with bounadry in W. Since the symplectic manifold V is an exact symplectic mainifold and W is an exact Lagrangian submanifold in V, by Stokes formula, we know that the possibility of $J-$holomorphic sphere or disk elimitated. So our priori assumption does not hold which implies the contact maifold (Σ, λ) has at least closed Reeb orbit. This finishes the proof of Theorem 1.1.

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